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Discussion

Further comments on the paper by Zinoviev and Bies, "On acoustic radiation by a rigid object in a fluid flow"

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Abstract

In this note, in response to the previous note by Zinoviev and Bies [Author's Reply to: F. Farassat, Comments on the paper by Zinoviev and Bies "On acoustic radiation by a rigid object in a fluid flow", *Journal of Sound and Vibration* 281 (2005) 1224–1237], the present authors briefly discuss the assumptions used in the acoustic analogy and its intended applications. It is pointed out that the scattering problems discussed by Zinoviev and Bies do not fall within the intended applications of the Ffowcs Williams–Hawkings (FW–H) equation. However, the FW–H equation can be used to derive an *integral equation* for finding scattered pressure fields. We derive this integral equation and show the validity of the equation for some of the examples of Zinoviev and Bies. We respond to the other issues brought up by these authors in their notes. We believe that Zinoviev and Bies have misinterpreted the acoustic analogy and have not applied the Curle formula and the FW–H equation correctly.

1. Introduction

It is unfortunate that in their response [1] to the letter to the Editor of this Journal by Farassat [2], Zinoviev and Bies have persisted in their claim that acoustic methodologies based on the Curle formula (CF) and the Ffowcs Williams–Hawkings (FW–H) equation do not give correct answers in some simple situations [3]. This claim has the potential of producing confusion in the minds of practicing acousticians and young researchers who use or are planning to use noise prediction methodologies based on the CF and FW–H equation. The present authors were, therefore, compelled to write this note to point out in detail the misinterpretation of the assumptions and the goals of the papers of Curle [4] and Ffowcs Williams and Hawkings [5], and the fallacy of the claim of Zinoviev and Bies. Erroneous conclusions like those expressed in Refs. [1,3] can be quite detrimental in research [6].

Zinoviev and Bies' response to Farassat [2] explains in detail relatively trivial matters of acoustics and fluid dynamics with reference citations. However, they have not carried out a thorough review of half a century of research on the acoustic analogy. Furthermore, they seem unwilling to admit that they have committed

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mathematical errors in dealing with the wave equation such as ignoring the retarded time in the manipulation of the integrand of the solution of the wave equation. In response to Farassat [2], they indicate that they have followed Curle's derivation, but this is untrue as will be shown below.

We will start by a discussion of the assumptions of the acoustic analogy and then respond to what we consider important points in the conclusions of Zinoviev and Bies' Reply [1].

2. The assumptions, applications and the current state of the acoustic analogy (AA)

The AA was developed more than half a century ago by Sir James Lighthill to study and explain jet noise generation [7,8]. Later Curle [4] and Ffowcs Williams and Hawkings [5] extended AA to the case of noise generation in the presence of stationary and moving surfaces, respectively. Let us state briefly the assumptions and applications of AA to establish a point of reference in refuting the claims of Zinoviev and Bies. Lighthill's assumptions, accepted also by Curle and Ffowcs Williams and Hawkings, are:

1. There is an agitated (e.g., turbulent) flow in a *finite* region of space which is generating noise.
2. *All* flow parameters (e.g., the Lighthill stress tensor and surface pressure) are known in the region of noise generation and reflection.
3. The noise generation process is not *sensitive* to small external disturbances (e.g., as in singing flames).

From the beginning, the AA was intended to be applied to some of *the most difficult problems of aeroacoustics for which no other noise prediction methodologies were available*. These problems are:

1. Noise prediction of stationary and moving jets.
2. Noise prediction of rotating blade propulsion machinery (e.g., propellers, turbofans and helicopter rotors) at all ranges of flight speed.
3. Airframe noise prediction for an aircraft at all ranges of flight speed.

To many people who are used to solving engineering and physics problems, the second assumption of Lighthill was (and still is) strange and troubling. There has been much discussion in the acoustics literature and at conferences as to the usefulness of AA particularly because of the second assumption. We will not belabor this point here other than to mention the following significant facts:

Fact 1. When Lighthill [7,8], Curle [4] and Ffowcs Williams and Hawkings [5] wrote their seminal papers, there were not much experimental or theoretical data to be used as input to their theories. Furthermore, there were no advanced digital computers available. For this reason, they derived qualitative results such as Lighthill's Eighth-Power Law of jet noise generation. Such results were obtained more than three decades ago and were quite useful in guiding many significant acoustic experiments and in designing low noise propulsion machinery. But AA has not been exclusively used for deriving qualitative results.

Fact 2. The rapid growth in high speed digital computer technology, the availability of turbulent flow simulation codes as well as high quality measured fluid dynamic data, and advances in the theory of partial differential equations, resulted in obtaining the needed data in AA for many of the important problems of aeroacoustics [9]. It is clear that today the usefulness of AA in applications outlined above is well established. In hindsight, Lighthill's AA was a stroke of genius from one of the giants of British applied mathematics which has been brought to fruition through the efforts of large groups of acoustic, fluid mechanics, mathematics and computer experts as well as talented experimentalists.

Fact 3. It is important to the aircraft industry that a noise prediction methodology be able to include *the exact geometry and kinematics* of the propulsion machinery. It has been our experience that *only a methodology based on the FW–H equation can satisfy these requirements*. The principal reason is the linearity of FW–H equation and the availability of many formally equivalent analytic solutions [5,10] from which the most appropriate formulation can be selected for a particular noise prediction problem. The division of work between the aerodynamicist to supply input data and the acoustician to predict the noise is a *strength* rather than a *weakness* of such a methodology. Any *fully CFD-based* computational aeroacoustics (CAA) methodology will be far too inefficient and beyond the capability of the supercomputers of today as compared to one based on the FW–H equation.

Fact 4. The mathematical machinery used in solving FW–H equation can also be used in many other problems of acoustics such as the scattering problems that Zinoviev and Bies consider [1,3]. If assumption 2 of Lighthill is relaxed, one can consider the resulting formal solution of the linear wave equation of FW–H as an integral equation when the observer is brought onto the data surface. The solution of this integral equation will then supply the missing input data to find the acoustic parameters off the data surface as predicted by the original FW–H equation. All linear steady and unsteady aerodynamics problems as well as many scattering problems can be solved by this approach [11,12]. It is the present authors' opinion that such an approach actually should *not* be considered as an application of the AA. The FW–H approach to such problems is more properly viewed as simply a device for obtaining integral representations of the solutions to boundary value problems.

Fact 5. *The wave equations of Lighthill, Curle, and Ffowcs Williams and Hawkings are exact! This means that if one solves these equations correctly for a problem satisfying the Lighthill assumptions, then one will get the correct answer to the acoustic problem.* To challenge this fact, as Zinoviev and Bies have done, is tantamount to the total mistrust of the usefulness of mathematics and physics in the scientific arena. Although this is a philosophical point, it is mentioned here because, in a deep sense, Zinoviev and Bies are claiming that AA, an exact theory based on rigorous and valid physics and mathematics, does not give the correct results for some problems. Given the exactness, and the demonstrated success, of the CF and FW–H theories, such a claim must be countered by asking the following questions:

(i) Is the problem formulated correctly so that it satisfies the Lighthill assumptions?
(ii) Is the problem solved correctly?

It is a maxim of the present authors that, on examining a problem under dispute, one or both of these questions will be answered negatively, thus refuting the claim of the failure of AA. This will be our approach here.

We will now address the points raised in Zinoviev and Bies' reply [1] to the present first author's comments [2] on the paper of Zinoviev and Bies [3].

3. The scattering problems and their solutions by AA

Strictly speaking, none of the problems considered by Zinoviev and Bies [1,3], was intended to be solved by AA. They are all trivial examples that are not in the category of the difficult aeroacoustics problems mentioned above. It is claimed by these authors that if one can show that AA gives the wrong answers for such simple problems, the accuracy of solutions for more advanced problems will be in question [1,3] in direct conflict with the current success of AA in acoustics [9,11–15]. In order to address this claim, we derive an integral equation for finding the scattered acoustic pressure based on the methodology of FW–H equation and then we will validate the result using some of the examples of Zinoviev and Bies.

In linear acoustics, the introduction of scattered pressure is for mathematical convenience. The scattered pressure is not a primary variable in experiments, i.e., it cannot be measured directly. What can be measured are the incident acoustic pressure (by the removal of the scatterer and turning the incident acoustic driver on) and the total acoustic pressure. To study the scattering problem using FW–H equation, we must first duplicate the theoretical approach of linear acoustics, i.e., obtain the equivalent normal velocity fluctuation and the scattered surface acoustic pressure on the scatterer. This is a trivial matter when we have no fluid flow as in the examples that Zinoviev and Bies consider. For such a stationary rigid scattering surface, we can use the following result:

$$v_n^s = -v_n^i, \tag{1}$$

where v_n is the local normal velocity of the scatterer. We will use superscripts s, i and t for scattered, incident and total acoustic pressures, respectively, everywhere in this note. We can now write the FW–H equation suitable for finding the scattered pressure as follows:

$$\Box^2 p^s = \rho_0 \dot{v}_n^s \delta(f) - \frac{\partial}{\partial x_i}[p^s n_i \delta(f)] = -\rho_0 \dot{v}_n^i \delta(f) - \frac{\partial}{\partial x_i}[p^s n_i \delta(f)] = \frac{\partial p^i}{\partial n}\delta(f) - \frac{\partial}{\partial x_i}[p^s n_i \delta(f)]. \tag{2}$$

We have assumed here that the scattering surface is described by $f(\mathbf{x}) = 0$, in such a way that $\nabla f = \mathbf{n}, \mathbf{n}$ being the unit outward normal to this surface. The symbol $\dot{v}_n$ stands for the time rate of change of v_n. In the last step of Eq. (2), we have used the linearized momentum equation to replace $-\rho_0 \dot{v}_n^i$ with $\partial p^i/\partial n$. We see that for a scattering problem, there is a *virtual normal velocity* v_n^s even if the surface is rigid. This just follows from the choice of mathematical approach.

We will now write the "formal solution" of Eq. (2) which is in fact an integral equation for finding the scattered acoustic pressure. It is

$$4\pi p^s(\mathbf{x}, t) = \int_{f=0} \frac{[\partial p^i/\partial n]_{\text{ret}}}{r}\,\mathrm{d}S + \int_{f=0}\left(\frac{[\dot{p}^s]_{\text{ret}}\cos\alpha}{c_0 r} + \frac{[p^s]_{\text{ret}}\cos\alpha}{r^2}\right)\mathrm{d}S. \tag{3}$$

Here, the angle α is between $\mathbf{n}$ and the radiation direction $\mathbf{r} = \mathbf{x} - \mathbf{y}$, where $\mathbf{y}$ is the source variable and c_0 is the speed of sound. If we assume here that the incident wave has the time dependence of the type $\mathrm{e}^{-\mathrm{i}\omega t}$, then the above integral equation can be written as

$$4\pi P^s(\mathbf{x}) = \int_{f=0} \frac{(\partial P^i/\partial n)\mathrm{e}^{\mathrm{i}kr}}{r}\,\mathrm{d}S + \int_{f=0} (-\mathrm{i}krP^s + P^s)\frac{\mathrm{e}^{\mathrm{i}kr}\cos\alpha}{r^2}\,\mathrm{d}S, \tag{4}$$

where $k = \omega/c_0$ is the wavenumber. We have now used capital letters for the complex amplitudes of the incident and the scattered acoustic pressure.

3.1. Example 1—scattering of a plane wave by a rigid sphere [3]

This is a classical scattering problem for which analytic solution can be obtained [16]. In the discussion of this problem, Zinoviev and Bies specified the surface pressure on the sphere from the available analytic solution in what they claimed to be an AA approach to the problem. Not obtaining the agreement with the analytic solution in the far field, they blamed the AA for giving the wrong answer. Because the incident acoustic pressure could be modeled by a single concentrated source far from the scattering surface, we used a different integral equation for this problem than Eq. (4) above [2]. The far-field form of the integral equation for the scattered pressure used was

$$4\pi P^s(\mathbf{x}) = -\mathrm{i}k \int_{f=0} \frac{P^t \mathrm{e}^{\mathrm{i}kr}\cos\alpha}{r}\,\mathrm{d}S, \tag{5}$$

where $P^t = P^i + P^s$ is the total acoustic pressure. We emphasize that this equation is valid *only for scattering by a closed rigid surface of the field incident from a point source exterior to the rigid surface* (see derivation in Ref. [2]). In the limit as the point source recedes to infinity, this also models an incident plane wave. A sign misprint in Eq. (11) of Ref. [2] is corrected on the right side of Eq. (5), above. We mention some other misprints in Ref. [2]. The speed of sound c_0 was dropped in the denominator of the second term in the integrand of Eq. (10), in Eq. (17) we have $A = -\mathrm{i}k\rho_0 c_0 U_0 R_0^2$ and a factor of 4π is missing on the right of Eq. (19).

We have shown that Eq. (5), which is based on the acoustic analogy, does give the correct scattered acoustic pressure for the problem considered here [2]. However, this equation *cannot* give the correct scattered pressure if the incident wave is a convergent spherical wave because such a wave cannot be modeled by a single concentrated source! Eq. (4), however, can always be used for such a problem. We will say some more in the next subsection about this.

We will first demonstrate the correctness of the integral equation (4) for the problem of scattering of a plane wave from a rigid sphere. We assume that the incident wave amplitude has the form

$P^i = \rho_0 c_0 U_0 e^{ikx_1} \equiv P_0 e^{ikx_1}$, i.e., the incident wave travels in the direction of the x_1-axis. In the far field, Eq. (4) is

$$4\pi P^s(\mathbf{x}) = \int_{f=0} \frac{(\partial P^i/\partial n)e^{ikr}}{r}\,dS - ik\int_{f=0} \frac{P^s e^{ikr}\cos\alpha}{r}\,dS. \tag{6}$$

We will again use spherical polar coordinates with the x_1-axis as the polar axis. As in Ref. [2], we use the observer and source variables $\mathbf{x} = (x, \Phi, \Theta)$ and $\mathbf{y} = (R_0, \varphi, \theta)$, respectively. We have $\partial P^i/\partial n = ikP^i$, and assuming that $kR_0 \ll 1$, the scattered acoustic pressure on the sphere is [16, p. 427, Eq. (9-1.8)]

$$P^s(R_0, \varphi, \theta) = \tfrac{1}{2} R_0 k P_0 \cos\theta. \tag{7}$$

For the observer in the far field, we can use the following relations:

$$\cos\alpha = \cos\theta\cos\Theta + \sin\theta\sin\Theta\cos(\Phi - \varphi), \tag{8}$$

$$|\mathbf{x} - \mathbf{y}| = x - R_0\cos\alpha, \tag{9}$$

$$\frac{e^{ikr}}{r} = \frac{e^{ikx}}{x}(1 - ikR_0\cos\alpha). \tag{10}$$

Using the above results in Eq. (6), we find

$$\begin{aligned} 4\pi P^s(x) &= ikR_0^2 P_0 \frac{e^{ikx}}{x}\int_0^{\pi}\int_0^{2\pi} e^{ikR_0\cos\theta}(1 - ikR_0\cos\alpha)\cos\theta\sin\theta\,d\varphi\,d\theta \\ &\quad + \frac{1}{2}k^2 R_0^3 P_0 \frac{e^{ikx}}{x}\int_0^{\pi}\int_0^{2\pi}(1 - ikR_0\cos\alpha)\cos\alpha\cos\theta\sin\theta\,d\varphi\,d\theta \\ &= -\frac{4\pi R_0^3}{3}k^2 P_0 \frac{e^{ikx}}{x}\left(1 - \frac{3}{2}\cos\theta\right) \quad \text{(far field)}, \end{aligned} \tag{11}$$

where we have used the relation $dS = R_0^2 \sin\theta\,d\varphi\,d\theta$ for the element of the surface area of the sphere in spherical polar coordinates. We have used *Mathematica* 5.1 to evaluate the above integrals. This is the correct result for the scattered acoustic pressure when the observer is in the far-field [16, p. 427, Eq. (9-1.8)]. Therefore, Eq. (4) which is based on the FW–H equation (actually equivalent to the Curle formula) does give the scattered acoustic pressure in the far field. We have now proven by two different integral equations based on the FW–H equation that AA gives the correct scattered acoustic pressure for a plane wave impinging on a rigid sphere.

The present first author in his comments on the work of Zinoviev and Bies [2] demonstrated that

1. These authors failed to model the source of the plane wave approaching the sphere from far field. See Eq. (6) of Ref. [2] and the discussion following it.
2. The authors failed to obtain the correct integral equation for the scattered acoustic pressure. Furthermore, they did not bring the derivatives with respect to the observer coordinates inside the integral to obtain a simple result. See Eq. (10) of Ref. [2].
3. The authors made numerous algebraic errors in manipulating the integral which gives the scattered acoustic pressure.

Once these errors were corrected by Farassat, *complete agreement of the scattered pressure from AA and analytic result was obtained* [2]. Zinoviev and Bies, however, in their reply to Farassat's comments on this problem [1], have insisted that "....this result is only fortuitous and the conclusion of Zinoviev and Bies that the *Curle formula does not describe properly the scattering of sound by a rigid object* is correct (italics in Ref. [1])". Such a response can only be described as astonishing in view of the various errors committed by these authors in the analysis of this problem. The correct response should have been that it is demonstrated that the *Curle formula as well as FW–H equation give correctly the scattering of sound by a rigid object.*

3.2. Example 2—a spherical wave converging on a rigid sphere

This example, for which a simple analytic solution can be obtained, was introduced by Zinoviev and Bies in Ref. [1, Section 3.2] as another problem where the application of AA results in an erroneous far field pressure. We will carry out a detailed analysis of this problem and we will demonstrate that Zinoviev's conclusion about AA giving the wrong result for this problem is erroneous. We will show that Eq. (5) above cannot be used for this problem as Zinoviev and Bies have done.

To generate the converging spherical wave, we need a spherical vibrating surface with large radius described by $f_\infty(\mathbf{x}) = 0$, where $\nabla f_\infty = \mathbf{n}$, $\mathbf{n}$ being the local inward facing unit normal to this surface. Now the FW–H equation for the problem under consideration is

$$\Box^2 p^t = \rho_0 \dot{v}_n \delta(f_\infty) - \frac{\partial}{\partial x_i}[p^t n_i \delta(f)] - \frac{\partial}{\partial x_i}[p^t n_i \delta(f_\infty)]. \tag{12}$$

We see that in this case, unlike the problem of scattering of a plane wave by a rigid sphere, we are not able to separate the incident and scattered wave because the incident wave is generated by a distributed source on a sphere. Compare the above Eq. (12) with Eq. (7) of Ref. [2] and note how the incident pressure contributions cancel from both sides of Eq. (7). Eq. (12) can now be used in several ways. First, it will lead to the correct integral equation for p^t on the surface $f = 0$, if we specify p^t and $\dot{v}_n$ on the surface $f_\infty(\mathbf{x}) = 0$. Second, if we specify p^t and $\dot{v}_n$ on both the inner and outer spheres (note $\dot{v}_n = 0$ on $f = 0$), then the solution of Eq. (12) will give p^t in the space between the two spheres. Neither of these options will give us the scattered pressure that we are looking for. However, the integral equation (4) and its far-field approximation, Eq. (6), based on FW–H equation are valid for this problem. We now prove the validity of Eq. (6).

The normal derivative of the incident acoustic pressure for this problem is [1]

$$\frac{\partial P^i(R_0, \varphi, \theta)}{\partial n} = -\frac{A(1 + ikR_0)e^{-ikR_0}}{R_0^2}, \tag{13}$$

where $P^i(x, \Phi, \Theta) = Ae^{-ikx}/x$. The scattered acoustic pressure can be shown to be [1]

$$P^s(x, \Phi, \Theta) = \frac{Be^{ikx}}{x}, \tag{14}$$

$$B = -Ae^{-2ikR_0}\frac{1 + ikR_0}{1 - ikR_0}. \tag{15}$$

Using Eqs. (8), (10), (13)–(15) in Eq. (6), we find that, in the far field

$$\begin{aligned}
&\int_{f=0} \frac{(\partial P^i/\partial n)e^{ikr}}{r}\,dS - ik\int_{f=0}\frac{P^s e^{ikr}\cos\alpha}{r}\,dS \\
&= -\frac{A(1 + ikR_0)e^{-ikR_0}}{R_0^2}\frac{e^{ikx}}{x}\int_0^\pi\int_0^{2\pi}(1 - ikR_0\cos\alpha)\sin\theta\,d\varphi\,d\theta \\
&\quad - \frac{ikBe^{ikR_0}}{R_0}\frac{e^{ikx}}{x}\int_0^\pi\int_0^{2\pi}(1 - ikR_0\cos\alpha)\cos\alpha\sin\theta\,d\varphi\,d\theta \\
&= -4\pi A\frac{1 + ikR_0}{1 - ikR_0}\frac{e^{ikx}}{x} + O(kR_0)^2 = 4\pi p^s(\mathbf{x}, t).
\end{aligned} \tag{16}$$

In the last step of this equation, we have utilized the assumption of $kR_0 \ll 1$. We have again used *Mathematica* 5.1 for evaluating the integrals and finding the Maclaurin series expansion of the result for small kR_0. Therefore, we have shown the validity of AA based on FW–H equation for this example also.

In summary, we note that Zinoviev and Bies have used the wrong integral equation (Eq. (5) above) for this problem. Eq. (5) was derived for a point source disturbance impinging on a rigid sphere [2]. For this reason their conclusion about FW–H equation giving the wrong result for this problem is not correct.

3.3. Example 3—sound generation by a sphere in a variable velocity field [1]

In Ref. [2], the present first author expressed the opinion that the origin of this problem was somewhat confusing as presented by Zinoviev and Bies in Ref. [3]. In his reply, Zinoviev has taken great pains to discuss the origin of this problem which is now stated clearly. See Section 3.3 of Ref. [1]. Having done this, Zinoviev goes on to show that, in his application of AA to this problem, based on the FW–H equation, one cannot obtain the correct and expected far field acoustic pressure [1,3]. We have a very simple explanation for the failure of Zinoviev and Bies' analysis of this problem:

> *The source of the generation of the variable unsteady flow over the sphere has not been modeled in the AA model of Zinoviev and Bies.* The contribution of this source to the acoustic pressure in the far field will add to that of Eq. (44) of Zinoviev and Bies [1] to give the expected result. This source is precisely what sets this problem apart from the problem of an oscillating (not vibrating, as the authors state in Ref. [1]) sphere in a stationary medium.

We will not carry out the lengthy analysis here based on the correct model of AA which includes the source of the generation of the variable unsteady flow over the sphere. In this problem also there is a *virtual normal velocity* on the sphere produced by the source of the variable unsteady flow. Thus, *the failure of AA as reported by Zinoviev and Bies is due to incorrect modeling in their AA analysis.*

3.4. Example 4—a rigid sphere embedded in a flow

This example is given in Section 3.4 of Ref. [1]. Here "a sphere is embedded in a velocity field in such a way that *it is stationary with respect to the fluid but moving with the fluid with respect to stationary observer* (italics in Ref. [1])". It is claimed by Zinoviev and Bies based on their interpretation of the FW–H equation that there is no radiation from the sphere in this case. This example is really a thought experiment that is incompletely described. The fluid and the sphere are assumed to be in motion while the observer is stationary with respect to the undisturbed medium. How is the fluid set into motion and how does the fluid velocity go to zero at the observer position? The present authors, therefore, see no reason to respond to the above claim of Zinoviev and Bies.

4. Other matters

One gets the impression from the response of Zinoviev and Bies [1] to Farassat [2] that these authors feel strongly that they have used mathematics correctly everywhere in their original paper [3]. We will not try to make this note a tutorial on mathematics. We will leave it to the authors to read about manipulation of solutions to the wave equation involving retarded time [16–21]. We will, however, first take issue with the definition of surface divergence.

4.1. The surface divergence

The divergence theorem is valid for a tangential vector field $\mathbf{V}$ on a smooth surface S as follows. Take a closed curve $\partial\sigma$ enclosing the piece of surface σ on S. Let $\mathbf{v}$ be the outward pointing unit geodesic normal to the curve $\partial\sigma$. A geodesic normal is a vector which is both tangent to S and normal to the curve $\partial\sigma$. Then, if $\nabla\cdot V$ is the surface divergence as defined by McConnell [22, p. 155, problem 5], we have the *divergence theorem*

$$\int_{\partial\sigma} V\cdot v\,\mathrm{d}l = \int_{\sigma} \nabla\cdot V\,\mathrm{d}\sigma. \tag{17}$$

Here $\mathrm{d}l$ is the element of length of $\partial\sigma$. As an example, we point out that it is with the above definition of surface divergence that Eq. (7.12) of Ref. [5] makes sense. We believe that the definition of surface divergence in Ref. [23, Eq. (5.6-4)] is meaningless and incorrect. Since this definition was originally utilized in the derivation of Eqs. (47) and (48) of Zinoviev and Bies [1] concerning the Curle formula, we will make further comments on the work of these authors on this formula below.

4.2. On the derivation of the Curle formula

We find the comments of Zinoviev and Bies on the work of Curle [4] in Section 4.3 of Ref. [1] confusing and confused. Rather than get involved in all the details of the algebraic manipulations leading to the Curle formula by the method that Curle himself used, we will make some important comments here related to his general approach.

1. The Curle formula was published half a century ago in 1955 using classical analysis. It is to be expected that the method used by Curle will be outdated by now. It is to Curle's credit that his formula, contrary to the claim of Zinoviev and Bies, is correct. Unfortunately, Zinoviev and Bies are using the same outdated mathematics, which leads to very complicated and tortuous algebraic manipulations, to try to show that Curle's formula is wrong.

2. N. Curle, a competent applied mathematician, was clearly aware which terms in the integrands of the integrals were evaluated at retarded time. See the statement following Eq. (2.4) of Ref. [4]. Also note that Curle [4] utilized Eq. (2.14) in his Eq. (2.13), which shows that, although Curle did not use the retarded time notation, he was obviously tracking which terms were evaluated at the retarded time. The omission of the retarded time notation in Curle's paper is unfortunate. But Curle was a pioneer in the field of aeroacoustics and his work was improved, validated and generalized by Ffowcs Williams and Hawkings [5].

3. We will now fill in the missing steps in Eq. (2.9) of Curle using [...] as the retarded time notation. Here we assume that $(\mathbf{y}, \tau)$ are the source space-time variables.

$$\begin{aligned}\frac{1}{r}\left[\frac{\partial^2 T_{ij}}{\partial^2 y_i y_j}\right] &= \frac{1}{r}\left(\frac{\partial}{\partial y_i}\left[\frac{\partial T_{ij}}{\partial y_j}\right] + \frac{\hat{r}_i}{c}\left[\frac{\partial^2 T_{ij}}{\partial \tau \partial y_j}\right]\right) = \frac{1}{r}\left(\frac{\partial}{\partial y_i}\left[\frac{\partial T_{ij}}{\partial y_j}\right] - \frac{\partial}{\partial x_i}\left[\frac{\partial T_{ij}}{\partial y_j}\right]\right) \\ &= \frac{\partial}{\partial y_i}\left(\frac{1}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right]\right) - \frac{\hat{r}_i}{r^2}\left[\frac{\partial T_{ij}}{\partial y_j}\right] - \frac{\partial}{\partial x_i}\left(\frac{1}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right]\right) + \frac{\hat{r}_i}{r^2}\left[\frac{\partial T_{ij}}{\partial y_j}\right] \\ &= \frac{\partial}{\partial y_i}\left(\frac{1}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right]\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right]\right),\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{1}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right] &= \frac{1}{r}\frac{\partial}{\partial y_j}[T_{ij}] - \frac{1}{r}\frac{\partial}{\partial x_j}[T_{ij}] \\ &= \frac{\partial}{\partial y_j}\left(\frac{1}{r}[T_{ij}]\right) - \frac{\hat{r}_i[T_{ij}]}{r^2} - \frac{\partial}{\partial x_j}\left(\frac{1}{r}[T_{ij}]\right) + \frac{\hat{r}_i[T_{ij}]}{r^2} \\ &= \frac{\partial}{\partial y_j}\left(\frac{1}{r}[T_{ij}]\right) - \frac{\partial}{\partial x_j}\left(\frac{1}{r}[T_{ij}]\right).\end{aligned} \tag{19}$$

In these equations, $\hat{\mathbf{r}}$ is the unit radiation vector $(\mathbf{x}-\mathbf{y})/r$. Using Eq. (19) in the last term on the right of Eq. (18), we get

$$\frac{1}{r}\left[\frac{\partial^2 T_{ij}}{\partial^2 y_i y_j}\right] = \frac{\partial}{\partial y_i}\left(\frac{1}{r}\left[\frac{\partial T_{ij}}{\partial y_j}\right]\right) + \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}[T_{ij}]\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial y_j}\left(\frac{1}{r}[T_{ij}]\right)\right). \tag{20}$$

This result was then used, after integration in variable $\mathbf{y}$ over a large volume and application of divergence theorem in this variable, to obtain the Curle Formula [4]. Of course, there are other terms in this formula which we have left out in the discussion here.

We give the above algebra to make several points. First, the notation of the retarded time is absolutely essential in getting the right result during the algebraic manipulations of the terms. Second, Curle has done the correct mathematics. Finally, we remind the readers that in the mid-fifties of the last century, it was very common, as Curle did, to assume that the readers of technical papers were quite capable of filling in the missing steps in algebra. In reading the response of Zinoviev and Bies [1] to Farassat [2], one gets the impression that they did not recognize that many algebraic steps were left out from Curle's paper [4]. For this reason, they felt that the retarded time effect was also neglected by Curle in his algebraic manipulation as they themselves did.

4. We believe that there are better ways of obtaining the Curle formula that avoid much of the algebra. For example, the Curle formula is obtained trivially from the FW–H equation, thus proving its validity. In working with the acoustics of moving surfaces, one finds that algebraic manipulations of the type of Eqs. (18)–(20), above, get very complicated and intractable when classical analysis is employed. One therefore requires new tools from advanced mathematics such as generalized function theory, partial differential equations and differential geometry. This is exactly what Ffowcs Williams and Hawkings did when they wrote their seminal paper in 1969 [5]. To understand and evaluate the work of Ffowcs Williams and Hawkings on AA, one needs to use the same mathematical tools as these authors have. Unfortunately, Zinoviev and Bies [1,3] not only have misinterpreted AA and applied the theory erroneously, they are also claiming that the available solutions of the FW–H equation need corrections or improvements. Zinoviev has even published a paper on this subject [24]. It cannot be overemphasized that the FW–H equation is an exact result and the available solutions [5,10,25,26] are neither in need of corrections nor any improvements.

5. The full account of contributions of the sources in the FW–H equation to the acoustic pressure, particularly the sources hidden in the Lighthill quadrupole term that come from the discontinuities in the fluid and the surface geometry, e.g., shocks, wakes and trailing edges, is given by Farassat, Brentner and Myers in Refs. [27–31]. We feel that the approach to AA that Ffowcs Williams and Hawkings expounded in their 1969 paper [5] has been the most fruitful approach to the subject thus far as evidenced by the enormous number of technical publications and the amount of scientific software used in the aircraft and engine industry based on the FW–H equation.

5. Concluding remarks

There are so many issues brought up by Zinoviev and Bies [1,3,24] that we felt that the current detailed response was necessary. We summarize our conclusions as follows:

1. The acoustic analogy was proposed for some of the most difficult aeroacoustic problems such as jet noise analysis. It was never intended for scattering problems although the approach of AA can be used to derive integral equations for the scattered pressure. In general, one can derive more than one integral equation, as we have shown. All these integral equations are equivalent.

2. We have shown repeatedly that Zinoviev and Bies have not applied AA correctly to the simple problems of scattering that they concocted. They appear to think that AA can be applied in recipé style (e.g., treating the local normal velocity of a rigid surface as zero in scattering problems) without adequate examination of the problem.

3. Zinoviev and Bies have made numerous errors in the solution of the wave equation. This has led them to apparently consequential, but incorrect, conclusions.

4. We see nothing in the work of Zinoviev and Bies to justify their disparaging claims about the Curle formula and the FW–H equation. Such unsubstantiated claims can damage the aeroacoustics community particularly by affecting the direction of research of new researchers who are not able to evaluate these authors' work.

Finally, we see no point in responding further to any erroneous claims of Zinoviev and Bies because this could only lead to an endless and fruitless dialogue.

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